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# A Framework for Best Simultaneous Approximation: Normed Almost Linear Spaces

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## 1. INTRODUCTION

Let $E$ be a normed linear space (nls) over the real field $R$ and $G$ a non-empty subset of $E$. For a bounded set $A \subset E$ let us define

$$\mathrm{rad}_G(A) = \inf_{g \in G} \sup_{a \in A} \|a - g\|, \tag{1.1}$$

$$\mathrm{cent}_G(A) = \{g_0 \in G: \sup_{a \in A} \|a - g_0\| = \mathrm{rad}_G(A)\}. \tag{1.2}$$

The number $\mathrm{rad}_G(A)$ is called the *Chebyshev radius* of $A$ with respect to $G$, and an element $g_0 \in \mathrm{cent}_G(A)$ is called a *best simultaneous approximation* (or a *Chebyshev center*) of $A$ with respect to $G$. When $A$ is a singleton, say $A = \{x\}, x \in E$, then $\mathrm{rad}_G(A)$ is the distance of $x$ to $G$, denoted by $\mathrm{dist}(x, G)$ and defined by

$$\mathrm{dist}(x, G) = \inf_{g \in G} \|x - g\| \tag{1.3}$$

and $\mathrm{cent}_G(A)$ is the set of all best approximations of $x$ out of $G$, denoted by $P_G(x)$ and defined by

$$P_G(x) = \{g_0 \in G: \|x - g_0\| = \mathrm{dist}(x, G)\}. \tag{1.4}$$

It is well known that for any bounded set $A \subset E$ we have

$$\mathrm{rad}_G(A) = \mathrm{rad}_G(\mathrm{co}\, A) = \mathrm{rad}_G(\bar{A}), \tag{1.5}$$

$$\mathrm{cent}_G(A) = \mathrm{cent}_G(\mathrm{co}\, A) = \mathrm{cent}_G(\bar{A}), \tag{1.6}$$

where co stands for the convex hull. Therefore the assumption on the bounded set $A$ to be convex (or (and) closed) is not a restrictive one.

A systematic study of the sets defined by (1.2) when $G = E$ was initiated



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by Garkavi [10–12] and the problem attracted much interest. At present many results are known on the existence and unicity of the elements of best simultaneous approximations, as well as on the continuity (semicontinuity) of $\mathrm{cent}_G$.

In this paper an attempt is made to obtain the natural framework for the theory of best simultaneous approximation in a nls, and we show that the normed almost linear space (nals) and the strong normed almost linear space (snals)—which are introduced in this paper—constitute this natural framework. These spaces, which might be of independent interest, generalize the normed linear spaces. To support the idea that the nals is a good concept, we introduce the concept of a "dual" space of a nals $X$, where the functionals on $X$ are no longer linear but "almost linear," which is also a nals. When $X$ is a nls, then the "dual" space defined by us is the usual dual space $X^*$.

Roughly speaking, a nals is a set $X$ together with two mappings $s: X \times X \to X$ and $m: R \times X \to X$, which satisfy some of the axioms of a linear space (ls), and on the set $X$ there exists a functional $|||\cdot|||: X \to R$ which satisfies all the axioms of an usual norm on a ls, as well as some additional ones (which in the case of a nls are consequences of the axioms of the norm). Denoting, as in the case of a ls, $s(x, y)$ and $m(\lambda, x)$, $(x, y \in X, \lambda \in R)$ by $x + y$ and $\lambda x$ (and also $m(-1, x)$ by $-x$), then for a subset $G \subset X$ we can define (1.3) and (1.4) replacing $\|\cdot\|$ by $|||\cdot|||$. Consequently, we can try to develop in a nals, a theory similar with that of the theory of best approximation in a nls.

Among the properties of a ls which is not supposed to hold in a nals $X$ is the following: for each $x \in X$ there exists $-x \in X$ such that $x + (-x) = 0$ (though there exists an element 0 in the nals $X$ such that $x + 0 = x$ for each $x \in X$). By the axioms of a nals, it follows that the set $V_X = \{x \in X: x + (-x) = 0\}$ is a ls, and we show that the theory of best simultaneous approximation in a nls is a particular case of the theory of best approximation in a nals $X$ by elements of subsets $G \subset V_X$.

In contrast with the case of a nls, the $|||\cdot|||$ of a nals $X$ does not generate a metric $\rho$ on $X$. (As a matter of fact, $\rho$ satisfies all the axioms of a metric, except for $\rho(x, x) = 0$ for each $x \in X$). Consequently, in a nals $X$ we cannot discuss the continuity (semicontinuity) properties of the (set-valued) mapping $x \to P_G(x)$. That is why we introduce the concept of a snals which, roughly speaking, is a nals $X$ together with a semi-metric $\rho$ on $X$ which is related in a certain way with the $|||\cdot|||$ of $X$.

In the framework of a nals (snals) we can try to extend the following types of results: (1) general results from the theory of normed linear spaces; (2) results from the theory of best approximation in a nls; (3) results from the theory of best simultaneous approximation in a nls. In this paper we begin such a study and we hope that other results will be extended.

We recall that another approach, completely different from ours, for the theory of best simultaneous approximation, was investigated in [23, 18, 9].

*All spaces involved in this paper are over the real field $R$.*

## 2. Normed Almost Linear Spaces

An *almost linear space* (als) is a set $X$ together with two mappings $s: X\times X \to X$ and $m: R\times X\to X$ satisfying $(L_1)$–$(L_8)$ below. For $x, y\in X$ and $\lambda\in R$ we denote $s(x, y)$ by $x+y$, $m(\lambda, x)$ by $\lambda x$ and $-1x$ by $-x$, when these will not lead to misunderstandings, and in the sequel $x-y$ means $x+(-y)$. Let $x, y z\in X$ and $\lambda, \mu\in R$. $(L_1)$ $(x+y)+z=x+(y+z)$; $(L_2)$ $x+y=y+x$; $(L_3)$ There exists an element $0\in X$ such that $x+0=x$ for each $x\in X$; $(L_4)$ $1x=x$; $(L_5)$ $0x=0$; $(L_6)$ $\lambda(x+y)=\lambda x+\lambda y$; $(L_7)$ $\lambda(\mu x)=(\lambda\mu)x$; $(L_8)$ $(\lambda+\mu)x=\lambda x+\mu x$ for $\lambda\geqslant 0$, $\mu\geqslant 0$.

2.1. *If $X$ is an als then*: (a) *The element* 0 *in $L_3$ is unique.* (b) $\lambda 0=0$ *for each $\lambda\in R$.* (c) *For each $x\in X$ and $\lambda\leqslant 0, \mu\leqslant 0, (\lambda+\mu)x=\lambda x+\mu x$.* (d) *If $x\in X$ is such that $x-x=0$, then $(\lambda+\mu)x=\lambda x+\mu x$ for all $\lambda, \mu\in R$.*

2.2. Definition. (a) A nonempty set $Y$ of an als $X$ is called an *almost linear subspace* of $X$, if for each $y_1, y_2\in Y$ and $\lambda\in R$, $s(y_1, y_2)\in Y$ and $m(\lambda, y_1)\in Y$. (b) An almost linear subspace $Y$ of $X$ is called a *linear subspace* of $X$ if $s: Y\times Y\to Y$ and $m: R\times Y\to Y$ satisfy all the axioms of a ls.

For an als $X$ we introduce the following sets:

$$V_X=\{x\in X: x-x=0\}, \tag{2.1}$$

$$W_X=\{x\in X: x=-x\}. \tag{2.2}$$

2.3. (a) *The set $V_X$ is a linear subspace of $X$, and it is the largest one. The als $X$ is a ls, iff $V_X=X$.* (b) *The set $W_X$ is an almost linear subspace of $X$, $W_X=\{x-x: x\in X\}$ and $V_X\cap W_X=\{0\}$. The als $X$ is a ls, iff $W_X=\{0\}$.*

Convex sets and cones in an als are defined as in a ls.

A *norm* on an als $X$ is a functional $|||\cdot|||: X\to R$ satisfying $(N_1$–$N_4)$ below. Let $x, y, z\in X$ and $\lambda\in R$. $(N_1)$ $|||x-z|||\leqslant |||x-y|||+|||y-z|||$; $(N_2)$ $|||\lambda x|||=|\lambda|\,|||x|||$; $(N_3)$ $|||x|||=0$ iff $x=0$. By $(N_1)$ we get

$$|||x+y|||\leqslant |||x|||+|||y|||. \tag{2.3}$$

By 2.3(a), $V_X$ is a ls and so $(V_X, |||\cdot|||)$ is a nls. Therefore the weak convergence (denoted by $\rightharpoonup$) can be defined in $V_X$. $(N_4)$ If $\{v_n\}_{n\in\Delta}$ is a net in $V_X$, $v\in V_X$, $v_n\rightharpoonup v$, then for each $x\in X$, $|||x-v|||\leqslant \liminf |||x-v_n|||$.

2.4. *For each* $x \in X$, $|||x||| \geqslant 0$ (*use* $(N_1)$ *and* $(N_2)$).

2.5. DEFINITION. An als $X$ together with $|||\cdot|||: X \to R$ satisfying $(N_1)$–$(N_4)$ is called a *normed almost linear space* (nals).

For $x \in X$ and $r > 0$ let $B_X(x, r) = \{y \in X: |||y - x||| \leqslant r\}$. Then $B_X(x, r)$ is a convex (possibly empty) subset of $X$. We denote $B_X = B_X(0, 1)$ and $S_X = \{x \in X: |||x||| = 1\}$.

2.6. *If $X$ is a nals, $X \neq V_X$, then $\rho(x, y) = |||x - y|||$, $x, y \in X$, is* not *a metric on $X$ since $\rho(x, x) \neq 0$ for $x \notin V_X$. By* 2.4, $(N_2)$, $(N_1)$, *and* 4.1 *in Section* 4, *$\rho$ satisfies all the other axioms of a metric.*

Our next aim is to introduce the concept of a dual space of a nals, which we would like to be a nals also.

2.7. DEFINITION. Let $X$ be an als. (a) A functional $f: X \to R$ is called an *almost linear functional* if (2.4)–(2.6) hold:

$$f(x + y) = f(x) + f(y) \qquad (x, y \in X), \tag{2.4}$$

$$f(\lambda x) = \lambda f(x) \qquad (\lambda \in R, \lambda \geqslant 0, x \in X), \tag{2.5}$$

$$-f(-x) \leqslant f(x) \qquad (x \in X). \tag{2.6}$$

(b) A functional $f: X \to R$ is called a *linear functional* if (2.4) and (2.5) hold for all $x, y \in X$ and $\lambda \in R$ (hence (2.6) is also satisfied).

Let $X^\#$ be the set of all almost linear functionals defined on the als $X$. For $f, f_1, f_2 \in X^\#$ and $\lambda \in R$ let $s(f_1, f_2)$ and $m(\lambda, f)$ be the functionals on $X$ defined by $s(f_1, f_2)(x) = f_1(x) + f_2(x)$ and $m(\lambda, f)(x) = f(\lambda x)$, $x \in X$. Then $s: X^\# \times X^\# \to X^\#$ and $m: R \times X^\# \to X^\#$ satisfy $(L_1)$–$(L_8)$, where $0 \in X^\#$ is the functional which is 0 at each $x \in X$. Therefore $X^\#$ is an als. Note that for each $f \in X^\#$, $f(0) = 0$ and the restriction $f \mid V_X$ is a linear functional on $V_X$. We shall denote $s(f_1, f_2)$ by $f_1 + f_2$ and $m(\lambda, f)$ by $\lambda \circ f$.

2.8. *Let* $f \in X^\#$. (a) *$f \in V_{X^\#}$ iff $f$ is linear on $X$, iff $-1 \circ f = -f$, iff $f \mid W_X = 0$;* (b) *The functional $h$ on $X$ defined by $h(x) = f(x) - f(-x)$, $x \in X$, belongs to $V_{X^\#}$;* (c) *If $f \notin V_{X^\#}$, then for the functional $h = f + (-1 \circ f) \in W_{X^\#}$ we have that $h \mid V_X = 0$ and $h \neq 0$.*

When $X$ is a nals, for $f \in X^\#$ define, as in the case of a nls,

$$|||f||| = \sup\{|f(x)|: x \in B_X\}. \tag{2.7}$$

Let $X^* = \{f \in X^\#: |||f||| < \infty\}$.

2.9. THEOREM. *$X^*$ together with $|||\cdot|||$ defined by (2.7) is a nals.*

*Proof.* It is easy to show that (2.3), ($N_2$) and ($N_3$) hold, so $X^*$ is an als. We show now $N_1$, i.e., that for $f_i \in X^*$, $i = 1, 2, 3$, we have

$$|||f_1 + (-1 \circ f_3)||| \leqslant |||f_1 + (-1 \circ f_2)||| + |||f_2 + (-1 \circ f_3)|||. \tag{2.8}$$

Let $x \in B_X$. Then $|(f_1 + (-1 \circ f_3))(x)| = |f_1(x) + f_3(-x)|$. If $|(f_1 + (-1 \circ f_3))(x)| = -f_1(x) - f_3(-x)$, then by (2.6) we get

$$\begin{aligned}|(f_1 + (-1 \circ f_3))(x)| &= -f_1(x) - f_3(-x)\\ &\leqslant f_1(-x) + f_2(x) + f_2(-x) + f_3(x)\\ &\leqslant |(f_1 + (-1 \circ f_2))(-x)| + |(f_2 + (-1 \circ f_3))(-x)|\\ &\leqslant |||f_1 + (-1 \circ f_2)||| + |||f_2 + (-1 \circ f_3)|||.\end{aligned}$$

Similarly we can show that the same conclusion holds if $|(f_1 + (-1 \circ f_3))(x)| = f_1(x) + f_3(-x)$, whence (2.8) follows.

It remains to prove ($N_4$). Let $(\varphi_n)_{n \in \Delta}$ be a net in $V_{X^*}$ and $\varphi_0 \in V_{X^*}$ such that $\varphi_n \rightharpoonup \varphi_0$. Let $x \in X$ and let $F_x$ be the functional on $V_{X^*}$ defined by $F_x(\varphi) = \varphi(x)$, $\varphi \in V_{X^*}$. By 2.9(a), $\varphi$ is a linear functional on $X$ and so $F_x$ is a linear functional on $V_{X^*}$. Since $|||F_x||| \leqslant |||x|||$, we have $F_x \in (V_{X^*})^*$. Since $\varphi_n \rightharpoonup \varphi_0$, it follows that $\lim \varphi_n(x) = \varphi_0(x)$. Let now $f \in X^*$ and $x \in B_X$. We have $|(f + (-1 \circ \varphi_0))(x)| = |f(x) + \varphi_0(-x)| = \lim |(f + (-1 \circ \varphi_n))(x)| \leqslant \liminf |||f + (-1 \circ \varphi_n)|||$, whence ($N_4$) follows.

2.10. DEFINITION. The space $X^*$ together with $|||\cdot|||$ defined by (2.7) is called the *dual* space of the nals $X$.

Clearly, when $X$ is a nls, then the dual space defined above is the usual dual space of $X$. That is why we did not change the notation and terminology.

2.11. DEFINITION. An almost lineat subspace $\Gamma$ of the dual space $X^*$ of a nals $X$ is said to be *total* over $X$ if the relations $x_1, x_2 \in X$, $f(x_1) = f(x_2)$ for each $f \in \Gamma$ imply that $x_1 = x_2$.

As we shall show by examples given in the next section, the dual space of a nals $X$ may be not total over $X$. As a matter of fact, we do not know whether $X^*$ might be only the single element 0.

For $x \in X$ let $Qx$ be the functional on $X^*$ defined, as in the case of a nls, by

$$Qx(f) = f(x) \qquad (f \in X^*). \tag{2.9}$$

Then $Qx$ is an almost linear functional on $X^*$ and $|||Qx||| \leqslant |||x|||$, i.e., $Qx \in X^{**}$. In contrast with the case of a nls, here $X^{**}$ (as well as $X^*$) may be very poor (see the next section).

We conclude this section by defining some subsets of a nals $X$ which will be used in Sections 4 and 5. For examples, see the next section.

2.12. *Let $X$ be a nals and let $\emptyset \neq G \subset V_X$. We define $R_X(G) \subset X$ in the following way: $x \in R_X(G)$ if for each $g \in G$ there exists $v_g \in V_X$ such that* (2.10) *and* (2.11) *hold*:

$$|||x-g||| = |||v_g - g|||, \tag{2.10}$$

$$|||x-v||| \geqslant |||v_g - v||| \qquad \textit{for each} \quad v \in V_X. \tag{2.11}$$

*We have $V_X \subset R_X(G)$. If $G_1 \subset G_2$, then $R_X(G_2) \subset R_X(G_1)$. We denote by $R_X$ the set $R_X(V_X)$. When $X$ is a nls, then $R_X = X$.*

## 3. Examples

In Examples 3.2, 3.4–3.8, and 3.11 we exhibit normed almost linear spaces which are not normed linear spaces. We recall the following definition (see, e.g., [24]).

3.1. Definition. Let $G$ be a nonempty subset of the nls $(E, \|\cdot\|)$ and let $C$ be a bounded subset of $E$. The set $C$ is called *remotal with respect to $G$* if for each $g \in G$ there exists $c_g \in C$ such that $\sup_{c \in C} \|c-g\| = \|c_g - g\|$.

3.2. Example. (a) Let $(E, \|\cdot\|)$ be a nls and let $X$ be the collection of all bounded, convex, nonempty subsets $A$ of $E$. For $A, A_1, A_2 \in X$ and $\lambda \in R$, define $s(A_1, A_2) = A_1 + A_2 = \{a_1 + a_2 : a_1 \in A_1, a_2 \in A_2\}$ and $m(\lambda, A) = \lambda A = \{\lambda a : a \in A\}$. The element 0 in $X$ is the set $\{0\}$. Then $X$ is an als and $V_X = \{\{v\} : v \in E\} (= E)$. (b) For $A \in X$, let $|||A||| = \sup_{a \in A} \|a\|$. It is straightforward that $|||\cdot||| : X \to R$ satisfies $(N_1)$–$(N_4)$ and so $X$ is a nals. (c) For each $\varphi \in E^*$ there is $f_\varphi \in X^*$, $|||f_\varphi||| = \|\varphi\|$. Indeed, take

$$f_\varphi(A) = \sup_{a \in A} \varphi(a) \qquad (A \in X). \tag{3.1}$$

For each $\varphi \in E^*$ there is $h_\varphi \in V_{X^*}$, $|||h_\varphi||| = \|\varphi\|$. Indeed, take $h_\varphi(A) = (f_\varphi(A) - f_\varphi(-A))/2$, $A \in X$, where $f_\varphi$ is defined by (3.1). Unfortunately we do not have a complete description of $X^*$ and $V_{X^*}$. Now let $A \in X$, where $A$ is remotal with respect to $G = \{0\}$. Then there exists $f \in S_{X^*}$ such that $f(A) = |||A|||$. Indeed, let $a_0 \in A$, $|||A||| = \|a_0\|$ and let $\varphi \in S_{E^*}$, $\varphi(a_0) = \|a_0\|$. Then for the functional $f_\varphi$ defined by (3.1) we have $|||f_\varphi||| = 1$ and

$f_\varphi(A) = |||A|||$. (d) Let $G$ be a nonempty subset of $E$. If $A \in X$ is remotal with respect to $G$, then $A \in R_X(G)$ (for $g \in G$ choose $v_g \in A$ with $\sup_{a \in A} \|a - g\| = \|v_g - g\|$; then $v_g$ satisfies (2.10) and (2.11)). The converse is not always true, as the following example shows.

3.3. EXAMPLE. Let $E = c$ (equals the Banach space of all convergent sequences endowed with the sup norm) and let $G = \{0\}$. Let $A \subset E$ be the set of all elements $a = (\alpha_n)$, where $\lim \alpha_n = 0$ and $|\alpha_n| \leqslant 1 - n^{-1}$, $n \in N$. Then $A \in X$ and $A$ is not remotal with respect to $G$. For the only element 0 in $G$ let $v_0 = (\beta_n) \in E$, where $\beta_n = 1 - n^{-1}$, $n \in N$. Then $|||A||| = \|v_0\| = 1$, and $|||A - v||| \geqslant \|v_0 - v\|$ for each $v \in E$, i.e., $A \in R_X(G)$.

3.4. EXAMPLE. (a) Let $(E, \|\cdot\|)$ be a nls and let $X$ be the collection of all bounded, closed, convex, nonempty subsets $A$ of $E$. Let $0 \in X$, $m$ and $|||\cdot|||$ be defined as in Example 3.2 and define $s(A_1, A_2) = \overline{A_1 + A_2}$, $A_1, A_2 \in X$. Then $X$ is a nals, and we can repeat word for word what was said in Example 3.2. (b) We can show that $X^*$ is total over $X$. Indeed, let $A_1, A_2 \in X$ such that $f(A_1) = f(A_2)$ for each $f \in X^*$. For $A \in X$ we have

$$A = \bigcap_{\varphi \in S_{E^*}} \{v \in E\colon \inf \varphi(A) \leqslant \varphi(v) \leqslant \sup \varphi(A)\}. \tag{3.2}$$

Now let $\varphi \in S_{E^*}$ and let $f_\varphi \in X^*$ be defined by (3.1). By hypothesis, we have $f_\varphi(A_1) = f_\varphi(A_2)$ and $(-1 \circ f_\varphi)(A_1) = (-1 \circ f_\varphi)(A_2)$, whence $\sup \varphi(A_1) = \sup \varphi(A_2)$ and $\inf \varphi(A_1) = \inf \varphi(A_2)$. By (3.2) we obtain that $A_1 = A_2$. Hence the mapping $Q\colon X \to X^{**}$ defined by (2.9) is injective. Here, we do not know an example of an $F \in X^{**}$, $F \neq Qx$ for each $x \in X$.

3.5. EXAMPLE. (a) Let $(E, \|\cdot\|)$ be a nls and let $X$ be the collection of all compact, convex, nonempty subsets of $E$. Let $s$, $m$, 0 and $|||\cdot|||$ be defined as in Example 3.4. Then $X$ is a nals which has all the properties of the nals discussed in Example 3.4. (b) For each $A \in X$ there exists $f \in S_{X^*}$ such that $f(A) = |||A|||$. Therefore in this space $|||QA||| = |||A|||$ for each $A \in X$, where $Q$ is defined by (2.9). (c) $R_X = X$.

3.6. EXAMPLE. (a) Let $(E, \|\cdot\|)$ be an (AL)-space (see, e.g., [7]) and let $X$ be the set $\{x \in E\colon x \geqslant 0\}$. For $x, y \in X$ and $\lambda \in R$, define $s(x, y) = x + y$ and $m(\lambda, x) = |\lambda|\, x$. The element 0 in $X$ is the element 0 in $E$. Then $X$ is an als and $V_X = \{0\}$. (b) For $x \in X$, let $|||x||| = \|x\|$. Then $|||\cdot|||$ satisfies $N_1 - N_4$, and so $X$ is a nals. (c) We have $X^* = \{\varphi \mid X\colon \varphi \in E^*, \varphi \geqslant 0\}$ and $V_{X^*} = \{0\}$. Here $X^*$ is total over $X$ and for each $x \in X$ there exists $f \in S_{X^*}$, with $f(x) = |||x|||$. (d) $R_X = \{0\}$.

3.7. EXAMPLE. (a) Let $E$ be a ls and $\varphi \in E^{\#}$, $\varphi \neq 0$. Let $X = \{x \in E: \varphi(x) > 0\} \cup \{0\}$. Define $s$, $m$ and 0 as in Example 3.6. Then $X$ is an als and $V_X = \{0\}$. (b) For $x \in X$, let $|||x||| = \varphi(x)$. We have $(N_1)$–$(N_4)$ and so $X$ together with $|||\cdot|||$ is a nals. (c) Let $f = \varphi \mid X$. We have $X^* = \{\lambda \circ f: \lambda \in R\} = \{\lambda f: \lambda \geqslant 0\}$ and $V_{X^*} = \{0\}$. For each $x \in X$ we have $f(x) = |||x|||$ and $|||f||| = 1$. (d) $R_X = \{0\}$.

3.8. EXAMPLE. (a) Let $(E, \|\cdot\|)$ be a nls and let $\varphi \in S_{E^*}$, $\varphi$ attains its norm on $S_E$. Then $H = \{x \in E: \varphi(x) = 0\}$ is proximinal in $E$ (i.e., $P_H(x) \neq \varnothing$ for each $x \in E$) and there exists a linear selection $p_H(x) \in P_H(x)$, $x \in X$ (see, e.g., [25]). Let $X = \{x \in E: \varphi(x) \geqslant 0\}$. For $x, y \in X$ and $\lambda \geqslant 0$ define $s(x, y) = x + y$, $m(\lambda, x) = \lambda x$ and $m(-1, x) = x - 2p_H(x)$. The element 0 in $X$ is the element $0 \in E$. Since $p_H$ is linear, $X$ is an als, and $V_X = H$. (b) Let $|||x||| = \varphi(x) + \|p_H(x)\|$, $x \in X$. Clearly, $|||\cdot|||$ satisfies $(N_2)$ and $(N_3)$. For $(N_1)$, let $x_i \in X$, $i = 1, 2, 3$. We have $|||x_1 + m(-1, x_3)||| = |||x_1 + x_3 - 2p_H(x_3)||| = \varphi(x_1) + \varphi(x_3) + \|p_H(x_1) - p_H(x_3)\| \leqslant \varphi(x_1 + x_2 - 2p_H(x_2)) + \|p_H(x_1 + x_2 - 2p_H(x_2))\| + \varphi(x_2 + x_3 - 2p_H(x_3)) + \|p_H(x_2 + x_3 - 2p_H(x_3))\| = |||x_1 + x_2 - 2p_H(x_2)||| + |||x_2 + x_3 - 2p_H(x_3)||| = |||x_1 + m(-1, x_2)||| + |||x_2 + m(-1, x_3)|||$. To show $(N_4)$, let $(v_n)_{n \in \Delta}$ be a net in $V_X (= H)$ and $v_0 \in V_X$ such that $v_n \rightharpoonup v_0$, and let $x \in X$. Then for each $n \in \Delta$ we have $|||x - v_0||| = \varphi(x - v_n) + \|p_H(x) - v_0\| \leqslant \varphi(x - v_n) + \liminf \|p_H(x) - v_n\| = \liminf(\varphi(x - v_n) + \|p_H(x) - v_n\|) = \liminf |||x - v_n|||$. Therefore $X$ is a nals. (c) Let $x_0 \in E$, $\varphi(x_0) > 0$ and $p_H(x_0) = 0$. We have $X^* = \{\psi \mid X: \psi \in E^*, \psi(x_0) \geqslant 0\}$ and $V_{X^*} = \{f \in X^*: f(x_0) = 0\}$. Here $X^*$ is total over $X$. (d) $R_X = X$. Indeed, let $x \in X$ and $g \in V_X$ $(= H)$. If $p_H(x) = g$, choose $v_g \in V_X$ such that $\|v_g - g\| = \varphi(x)$. If $p_H(x) \neq g$, let $v_g = \lambda p_H(x) + (1 - \lambda) g \in V_X$, where $\lambda = 1 + \varphi(x)/\|p_H(x) - g\|$. In both cases $v_g$ satisfies (2.10) and (2.11).

Lima introduced and studied the notion of a semi $L$-summand in a Banach space. We recall the definition [19, Sect. 5].

3.9. DEFINITION. A linear subspace $G$ of a Banach space $E$ is *a semi-L-summand* in $E$ if $G$ is Chebyshev in $E$ (i.e., $P_G(x)$ is a singleton for each $x \in E$) and $\|x\| = \|x - P_G(x)\| + \|P_G(x)\|$, $x \in E$.

3.10. (a) *Let $(E, \|\cdot\|)$ be a Banach space and $G$ a semi L-summand in $E$. Let $X$ be a convex cone with vertex at $0 \in E$ such that $G \subset X$ and for each $x \in X \backslash G$, $-x \notin X$. Let us also assume that $P_G$ is additive on $X$. Define $s$, $m$, and* 0 *as in Example* 3.8 *where we replace $p_H$ by $P_G$. Then $X$ is an als and $V_X = G$.* (b) *For $x \in X$ let $|||x||| = \|x\|$. Then $|||\cdot|||$ satisfies* $(N_2)$–$(N_4)$.

Let $C(Q)$ be the Banach space of all continuous functions over the compact Hausdorff space $Q$ endowed with the sup norm.

3.11. EXAMPLE. (a) Let $E=C([\alpha, \beta])$, $G$ the subspace of $E$ consisting of all functions which are constant on $[\alpha, \beta]$ and $X \subset E$ the set of all nondecreasing (similarly for nonincreasing) continuous functions on $[\alpha, \beta]$. By [19, Theorem 7.8] $G$ is a semi-$L$-summand in $E$. The set $X$ is a convex cone with vertex at $0 \in E$, $G \subset X$ and for $x \in X \setminus G$ we have $-x \notin X$. If $x \in X$, then $P_G(x)$ is the constant function on $[\alpha, \beta]$ which equals $(x(\beta)+x(\alpha))/2$. Then it is obvious that $P_G$ is additive on $X$. (We point out here that $P_G$ is not additive on $E$ (see, e.g., [15, p. 163]). Let $s$, $m$, and 0 be defined as in 3.10(a). As we have observed there, $X$ is an als and $V_X=G$. (b) Let $|||x|||=\|x\|$ for each $x \in X$. By 3.10(b), $|||\cdot|||$ satisfies $(N_2)$–$(N_4)$. Now we show that it satisfies also $(N_1)$. Let $x_i \in X$, $i=1, 2, 3$. Since for $x \in X$, $P_G(x)$ is the constant function $(x(\beta)+x(\alpha))/2$ and $\|x\|=\max\{|x(\alpha)|, |x(\beta)|\}$ we get $|||x_1+m(-1, x_3)|||=\|x_1+x_3-2P_G(x_3)\|=\max\{|x_1(\alpha)-x_3(\beta)|, |x_1(\beta)-x_3(\alpha)|\}$. Suppose we have

$$|x_1(\alpha)-x_3(\beta)| \leqslant |x_1(\beta)-x_3(\alpha)|. \tag{3.3}$$

If $|||x_1+m(-1, x_3)|||=x_1(\beta)-x_3(\alpha)$, then since $x_2(\alpha) \leqslant x_2(\beta)$, we have $|||x_1+m(-1, x_3)||| \leqslant x_1(\beta)-x_2(\alpha)+x_2(\beta)-x_3(\alpha) \leqslant |x_1(\beta)-x_2(\alpha)| + |x_2(\beta)-x_3(\alpha)| \leqslant |||x_1+m(-1, x_2)|||+|||x_2+m(-1, x_3)|||$. If $|||x_1+m(-1, x_3)|||=x_3(\alpha)-x_1(\beta)$, by (3.3) and since $x_i(\alpha) \leqslant x_i(\beta)$, $i=1, 3$, we have $-x_1(\alpha)+x_3(\beta) \leqslant x_3(\alpha)-x_1(\beta)$, whence it follows $x_1(\alpha)=x_1(\beta)$ and $x_3(\alpha)=x_3(\beta)$. Then as above, $|||x_1+m(-1, x_3)|||=x_3(\beta)-x_1(\alpha) \leqslant |||x_1+m(-1, x_2)|||+|||x_2+m(-1, x_3)|||$. The case $|x_1(\beta)-x_3(\alpha)| \leqslant |x_1(\alpha)-x_3(\beta)|$ is proved in a similar way. Consequently $|||\cdot|||$ satisfies $(N_1)$. Therefore $X$ is a nals. (c) For $\alpha \leqslant \gamma \leqslant \beta$ define the functional $f_\gamma$ by $f_\gamma(x)=(x(\beta)+x(\gamma))/2$, $x \in X$. Then $f_\gamma \in X^*$ and $f_\alpha \in V_{X^*}$. Note that $X^*$ is total over $X$ and for each $x \in X$ there is $f \in S_{X^*}$ such that $f(x)=|||x|||$. (d) $R_X=X$. Indeed, let $g \in V_X$ $(=G)$ and $x \in X$. We denote by $g$ the value of the constant function $g$ on $[\alpha, \beta]$. If $|x(\alpha)-g| \leqslant |x(\beta)-g|$ (resp. $|x(\beta)-g| \leqslant |x(\alpha)-g|$), take $v_g \in V_X$ the constant function which equals $x(\beta)$ (resp. $x(\alpha)$). Then $v_g$ satisfies (2.10) and (2.11).

## 4. BEST APPROXIMATION IN NORMED ALMOST LINEAR SPACES

Let $X$ be a nals, $G$ a subset of $X$, and $x \in X$. We define $\operatorname{dist}(x, G)$ and $P_G(x)$ by (1.3) and (1.4), where we replace $\|\cdot\|$ by $|||\cdot|||$, keeping the same definitions for proximinal and Chebyshev sets as in a nls. We denote $\operatorname{Dom}(P_G)=\{x \in X: P_G(x) \neq \varnothing\}$.

Some simple properties of the function $\mathrm{dist}(\cdot, G)$ and of the set $P_G(x)$ from the theory of best approximation in a nls can be extended in a nals with similar proofs. Some other results from the theory of best approximation in a nls can be proved in a similar way for a nals, but not in the whole generality. One of the difficulties which appear when we want to extend for a nals $X$ some results which hold in a nls is due to the fact that the function $\rho(x, y) = |||x - y|||$, $x, y \in X$, is not a metric on $X$. Consequently, for some elements $g \in G$ we can get $\mathrm{dist}(g, G) \neq 0$ and $P_G(g) = \emptyset$, as simple examples show.

4.1. *If* $\mathrm{dist}(x, G) = 0$ *for some* $x \in X$, *then* $x \in V_X$. *Indeed, let* $\{g_n\}_{n=1}^{\infty} \subset G$ *such that* $|||x - g_n||| \to 0$. *Then* $|||x - x||| \leqslant |||x - g_n||| + |||g_n - x||| = 2\,|||x - g_n|||$ *for each n, and so* $|||x - x||| = 0$, *whence* $x - x = 0$, *i.e.*, $x \in V_X$.

Since it is very difficult to obtain results when $G$ is an arbitrary subset of $X$, as a first step in the theory of best approximation in a nals *we shall consider in this section only the case when* $G \subset V_X$. The restriction to subsets $G \subset V_X$ is of course severe, but we note that in Example 3.2 (similarly for Examples 3.4 and 3.5) if $x \in X$ stands for the bounded, convex, nonempty set $A \subset E$, then for any $G \subset V_X$ $(= E)$ we have

$$\mathrm{dist}(x, G) = \mathrm{rad}_G(A), \tag{4.1}$$

$$P_G(x) = \mathrm{cent}_G(A). \tag{4.2}$$

Consequently, any information we get on the function $\mathrm{dist}(\cdot, G)$ and on the set-valued mapping $x \to P_G(x)$, when $G \subset V_X$ and $X$ is a nals, are also valid for the function $\mathrm{rad}_G(\cdot)$ and for the set-valued mapping $A \to \mathrm{cent}_G(A)$, $A$ a bounded, nonempty subset of $E$ (in view of (1.5) and (1.6)). Therefore the theory of best simultaneous approximation in a nls is a particular case of the theory of best approximation in a nals by elements of subsets $G \subset V_X$.

When $G \subset V_X$, then many more notions and results from the theory of best approximation in a nls can be formulated and proved in a similar way for a nals, e.g., [25, Chap. 1, Theorems 6.1 and 6.5], all assertions which do not involve a topology [8], the approximatively compact sets with the consequence that such sets are proximinal [26, Propositions 2.1 and 3.1], $1° \Leftrightarrow 2°$ [1, Proposition 4.1]. We shall refer several times to the following immediate (due to $(N_4)$) result.

4.2. Proposition. *Let $X$ be a nals and $G$ a boundedly weakly compact subset of $V_X$. Then $G$ is proximinal in $X$.*

Yost [28] introduced and studied the closed linear subspaces with the

$1\frac{1}{2}$-ball property in a Banach space $E$ and proved that they are proximinal in $E$. The next proposition is a localization of this result for a nals.

Let $X$ be a nals and $\emptyset \neq G \subset V_X$. Let $T_G$ be the subset of $X$ defined in the following way: $x \in T_G$ if for each $g \in G$ and $r_i > 0$, $i = 1, 2$, the relations $|||x - g||| < r_1 + r_2$, $B_X(x, r_2) \cap G \neq \emptyset$ imply that $B_X(g, r_1) \cap B_X(x, r_2) \cap G \neq \emptyset$. Clearly we have $G \subset T_G$.

4.3. PROPOSITION. *Let $X$ be a nals and $G$ a complete subset of $V_X$. Then for each $x \in T_G$ we have $P_G(x) \neq \emptyset$.*

It follows from [27] that $\operatorname{cent}_G(A) \neq \emptyset$ for every bounded nonempty subset $A \subset C(Q)$, where $G = \{x \in C(Q): x \mid K = 0\}$, and $K$ is a closed subset of $Q$. The next result shows that for compact (convex) sets of $C(Q)$ we have (due to Proposition 4.3) a stronger property, namely, that they belong to $T_G$ (when $X$ is the nals described in Example 3.2 for $E = C(Q)$). The fact that the sets containing exactly one point belong to $T_G$ follows from [28].

4.4. THEOREM. *Let $E = C(Q)$ and for $K \subset Q$, $K$ closed, let $G = \{x \in C(Q): x \mid K = 0\}$. Then for each compact set $A \subset E$ the relations $g \in G$, $g_1 \in G$, $r_1, r_2 > 0$, $\|x - g\| < r_1 + r_2$ for each $x \in A$ and $\|x - g_1\| \leqslant r_2$ for each $x \in A$ imply that there exists $g_0 \in G$ such that $\|g - g_0\| \leqslant r_1$ and $\|x - g_0\| \leqslant r_2$ for each $x \in A$.*

*Proof.* Let $H(R) = \{[a, b]: a, b \in R, a \leqslant b\}$, and for $q \in Q$ let $\psi(q)$ be defined by

$$\psi(q) = \bigcap_{x \in A} [x(q) - r_2, x(q) + r_2] \cap [g(q) - r_1, g(q) + r_1]. \tag{4.3}$$

We show that $\psi: Q \to H(R)$. Clearly, $\psi(q)$ is closed and convex and so it remains to show that $\psi(q) \neq \emptyset$. By hypothesis, $\|x - g\| < r_1 + r_2$ for each $x \in A$ and so $|x(q) - g(q)| < r_1 + r_2$ for each $x \in A$. Hence, we get

$$x_1(q) = \sup_{x \in A}(x(q) - r_2) < g(q) + r_1, \tag{4.4}$$

$$x_2(q) = \inf_{x \in A}(x(q) + r_2) > g(q) - r_1. \tag{4.5}$$

Then $x_1(q) \leqslant x_2(q)$ since otherwise, by (4.4) and (4.5) there exist $x, y \in A$ such that $y(q) + r_2 < x(q) - r_2$ and so $2r_2 < x(q) - y(q) = (x(q) - g_1(q)) + (g_1(q) - y(q)) \leqslant \|x - g_1\| + \|y - g_1\| \leqslant 2r_2$, which is not possible. Then

$$\psi(q) = [x_1(q), x_2(q)] \cap [g(q) - r_1, g(q) + r_1] \tag{4.6}$$

and by (4.4) and (4.5) it follows $\psi(q) \neq \emptyset$.

We show now that $\psi: Q \to H(R)$ is lower semicontinuous (lsc) (for the definition of lsc see, e.g., [25]). Let $q_0 \in Q$ and $(c, d)$ an open interval such that $\psi(q_0) \cap (c, d) \neq \emptyset$. Let $\alpha_0 \in \psi(q_0) \cap (c, d)$ and let $0 < \varepsilon < \min\{\alpha_0 - c, d - \alpha_0\}$. Since $A$ is compact, the functions $x_1$ and $x_2$ defined by (4.4) and (4.5) are continuous. There exists a neighbourhood $U(q_0)$ such that

$$|x_1(q) - x_1(q_0)| < \varepsilon, \qquad q \in U(q_0), \tag{4.7}$$

$$|x_2(q) - x_2(q_0)| < \varepsilon, \qquad q \in U(q_0), \tag{4.8}$$

$$|g(q) - g(q_0)| < \varepsilon, \qquad q \in U(q_0). \tag{4.9}$$

We show that $\psi(q) \cap (c, d) \neq \emptyset$ for each $q \in U(q_0)$ which will prove that $\psi$ is lsc. Suppose $\psi(q) \cap (c, d) = \emptyset$ for some $q \in U(q_0)$. Then we have either $\alpha \leqslant c$ for each $\alpha \in \psi(q)$ or $d \leqslant \alpha$ for each $\alpha \in \psi(q)$, say the former (the proof for the latter case is similar). Then by (4.6) we have either $x_2(q) \in \psi(q)$ or $g(q) + r_1 \in \psi(q)$. If $x_2(q) \in \psi(q)$, then by our assumption $x_2(q) \leqslant c$, and by (4.8) we get $x_2(q_0) < x_2(q) + \varepsilon \leqslant c + \varepsilon < c + (\alpha_0 - c) = \alpha_0$, which is not possible since $\alpha_0 \in \psi(q_0)$. If $g(q) + r_1 \in \psi(q)$, then $g(q_0) + r_1 < g(q) + r_1 + \varepsilon \leqslant c + \varepsilon < \alpha_0$ which is not possible since $\alpha_0 \in \psi(q_0)$.

If $q \in K$, then $0 \in \psi(q)$. Indeed, if $q \in K$ then for each $x \in A$ we have $|x(q)| = |x(q) - g_1(q)| \leqslant \|x - g_1\| \leqslant r_2$, and so $-r_2 \leqslant x(q) \leqslant r_2$. Then by (4.4) and (4.5) we get $x_1(q) \leqslant 0$ and $x_2(q) \geqslant 0$, whence $0 \in \psi(q)$ follows now by (4.6) since $g(q) = 0$.

Define $\eta: Q \to H(R)$ by

$$\eta(q) = \begin{cases} \psi(q), & q \notin K, \\ \{0\}, & q \in K. \end{cases}$$

Then $\eta$ is lsc and by Michael's theorem [22] there exists $g_0 \in C(Q)$ such that $g_0(q) \in \eta(q)$, $q \in Q$. For $q \in K$, $g_0(q) = 0$, i.e., $g_0 \in G$. Since for each $q \in Q$, $g_0(q) \in \psi(q)$, by (4.3) we get $\|g - g_0\| \leqslant r_1$ and $\|x - g_0\| \leqslant r_2$ for each $x \in A$, which completes the proof.

Let us recall the following definition [2, Definition 1.1].

4.5. Definition. The nls $E$ is said to be *strictly convex with respect to its linear subspace* $G$ if the conditions $x, y \in E$, $\|x\| = \|y\| = \|(x + y)/2\| = 1$, $x - y \in G$ imply that $x = y$.

In the results 4.6–4.9 we shall use the set $R_X(G)$ defined in 2.12. The next theorem generalizes results on best simultaneous approximation contained in [15, p. 188; 4, 2, 24].

4.6. Theorem. *Let $X$ be a nals and $G$ a linear subspace of $V_X$. If $V_X$ is strictly convex with respect to $G$, then for each $x \in R_X(G)$ the set $P_G(x)$ con-*

*tains at most one element. If in addition $G$ is reflexive then for each $x \in R_X(G)$ the set $P_G(x)$ is a singleton.*

*Proof.* Let $x \in R_X(G)$ and suppose there exist $g_1, g_2 \in G$ such that $|||x - g_i||| = \text{dist}(x, G)$, $i = 1, 2$. Then $|||x - (g_1 + g_2)/2||| = \text{dist}(x, G)$. Since $x \in R_X(G)$, for the element $(g_1 + g_2)/2 \in G$ there exists $v_0 \in V_X$ such that $|||x - (g_1 + g_2)/2||| = |||v_0 - (g_1 + g_2)/2|||$, and $|||x - g_i||| \geqslant |||v_0 - g_i|||$, $i = 1, 2$. Then $\text{dist}(x, G) = |||v_0 - (g_1 + g_2)/2||| \leqslant (|||v_0 - g_1||| + |||v_0 - g_2|||)/2 \leqslant \text{dist}(x, G)$, and so $|||v_0 - g_1||| = |||v_0 - g_2||| = |||v_0 - (g_1 + g_2)/2|||$. Since $(v_0 - g_1) - (v_0 - g_2) = g_2 - g_1 \in G$ and $V_X$ is strictly convex with respect to $G$, it follows that $g_1 = g_2$. If $G$ is reflexive then by Proposition 4.2, $G$ is proximinal in $X$, which completes the proof.

Let $X$ be a nals and $\varnothing \neq G \subset V_X$. We shall assign to each $x \in R_X(G)$ a nonempty subset $D_G(x) \subset V_X$ in the following way. For $g \in G$ let $D_g(x)$ be the set of all $v_g \in V_X$ satisfying (2.10) and (2.11). Since $x \in R_X(G)$, the set $D_g(x)$ is nonempty. Let us set

$$D_G(x) = \bigcup_{g \in G} D_g(x), \qquad D(x) = D_{V_X}(x).$$

4.7. LEMMA. *Let $x \in R_X(G)$ and $g \in G$. Then for each $v_g \in D_g(x)$ we have*

$$|||x - g||| = |||v_g - g||| = \sup\nolimits_{y \in D_G(x)} |||y - g|||. \tag{4.10}$$

*Consequently, the set $D_G(x)$ is a nonempty, bounded subset of $V_X$, which is remotal with respect to $G$. If $x \in V_X$, then $D_G(x) = \{x\}$.*

*Proof.* Let $x \in R_X(G)$, $g \in G$ and $v_g \in D_g(x)$. By (2.10) we have $|||x - g||| = |||v_g - g|||$. Let $y \in D_G(x)$. By (2.11) we have $|||x - g||| \geqslant |||y - g|||$. Therefore $|||v_g - g||| = |||x - g||| \geqslant |||y - g|||$, whence (4.10) follows since $v_g \in D_G(x)$. Let now $x \in V_X$ $(\subset R_X(G))$ and $v_0 \in D_G(x)$. By (2.11) for $v = x \in V_X$ we have $0 = |||x - x||| \geqslant |||v_0 - x|||$, i.e., $x = v_0$.

4.8. THEOREM. *Let $X$ be a nals, $\varnothing \neq G_1 \subset G \subset V_X$ and let $x \in R_X(G)$. We have*

$$\text{dist}(x, G_1) = \text{rad}_{G_1}(D_G(x)), \tag{4.11}$$

$$P_{G_1}(x) = \text{cent}_{G_1}(D_G(x)). \tag{4.12}$$

*Proof.* Let $g_1 \in G_1$. Since $x \in R_X(G)$ and $G_1 \subset G$, by Lemma 4.7 we have

$$|||x - g_1||| = \sup\nolimits_{y \in D_G(x)} |||y - g_1|||,$$

whence (4.11) follows by taking the infimum in both sides over all $g_1 \in G_1$. The proof of (4.12) is an immediate consequence of (4.10) and (4.11).

4.9. COROLLARY. *Let $X$ be a nals and $\varnothing \neq G \subset V_X$ such that $R_X(G) = X$. Suppose that for every nonempty bounded set $A \subset V_X$ we have* $\operatorname{cent}_G(A) \neq \varnothing$ ($\operatorname{cent}_G(A)$ *a singleton*). *Then $G$ is proximal* (*Chebyshev*) *in $X$.*

Borwein and Keener [3] examined the relationship between the Hausdorff distance between convex closed sets in a nls and the distance between their Chebyshev centers. Now we can assign to any $x \in R_X$ the set $\overline{\text{co}}\, D(x)$ and by (1.5) and (1.6), formulas (4.11) and (4.12) hold for $\overline{\text{co}}\, D(x)$ as well as for $D(x)$. Then some consideration of [3] can be used together with Theorem 4.8, to obtain information for a nals $X$.

We conclude this section by introducing the quotient space $X/G$, where $X$ is a nals and $G$ a closed linear subspace of $V_X$.

First, let $X$ be an als and $G$ a linear subspace of $V_X$, and let $X/G = \{\hat{x} = x + G: x \in X\}$. Clearly $\hat{x} = \hat{y}$ iff $x = y + g$ for some $g \in G$. (Simple examples show that this is no longer true when $G$ is an arbitrary almost linear subspace of $X$.) As in the case when $X$ is a ls, we can define $s(\hat{x}_1, \hat{x}_2)$, $\hat{0}$ and $m(\lambda, \hat{x})$. It is easy to show that $X/G$ together with $s: X/G \times X/G \to X/G$ and $m: R \times X/G \to X/G$ is an als. Here we have $V_{X/G} = V_X/G$.

Now suppose $X$ is a nals and $G$ a linear subspace of $V_X$. Let us define for $\hat{x} \in X/G$,

$$|||\hat{x}||| = \operatorname{dist}(x, G), \tag{4.13}$$

where $x \in \hat{x}$. Since $\operatorname{dist}(x + g, G) = \operatorname{dist}(x, G)$, $g \in G$, $|||\hat{x}|||$ does not depend on the choice of $x \in \hat{x}$.

4.10. THEOREM. *Let $X$ be a* nals *and $G$ a closed linear subspace of $V_X$. Then $X/G$ is an* als *and* $|||\cdot|||$ *defined by* (4.13) *satisfies* ($N_1$)–($N_3$). *It satisfies* ($N_4$) *if* $\dim V_X/G < \infty$ *or $V_X$ is reflexive.*

*Proof.* As we have observed above, $X/G$ is an als, and it is easy to prove that ($N_2$) and ($N_3$) hold. To show ($N_1$), let $\hat{x}_i \in X/G$, $x_i \in \hat{x}_i$, $i = 1, 2, 3$, and let $\varepsilon > 0$. By (4.13) there exist $g_1, g_2 \in G$ with $|||x_1 - x_3 - g_1||| \leq \operatorname{dist}(x_1 - x_3, G) + \varepsilon$ and $|||x_3 - x_2 - g_2||| \leq \operatorname{dist}(x_3 - x_2, G) + \varepsilon$. Then $|||\hat{x}_1 - \hat{x}_2||| = \operatorname{dist}(x_1 - x_2, G) \leq |||(x_1 - g_1) - (x_2 + g_2)||| \leq |||x_1 - g_1 - x_3||| + |||x_3 - x_2 - g_2||| \leq \operatorname{dist}(x_1 - x_3, G) + \operatorname{dist}(x_3 - x_2, G) + 2\varepsilon = |||\hat{x}_1 - \hat{x}_3||| + |||\hat{x}_3 - \hat{x}_2||| + 2\varepsilon$, whence ($N_1$) follows.

To show ($N_4$), let $\{\hat{v}_n\}_{n \in \Delta \cup \{0\}}$ be a net in $V_X/G$ ($= V_{X/G}$) such that $\hat{v}_n \rightharpoonup \hat{v}_0$, and suppose there exists $\hat{x} \in X/G$ such that $\liminf |||\hat{x} - \hat{v}_n||| < |||\hat{x} - \hat{v}_0|||$. We can suppose (passing to a subnet) that $|||\hat{x} - \hat{v}_n||| < \alpha < |||\hat{x} - \hat{v}_0|||$, $n \in \Delta$. Let $x \in \hat{x}$ and $v_n \in \hat{v}_n$, $n \in \Delta \cup \{0\}$. Then by (4.13) there exist $g_n \in G$, $n \in \Delta$ such that

$$|||x - v_n - g_n||| < \alpha < \operatorname{dist}(x - v_0, G), \qquad n \in \Delta. \tag{4.14}$$

Suppose $\dim V_X/G < \infty$. Then $\hat{v}_n \to \hat{v}_0$ and so there exist $\bar{g}_n \in G$, $n \in \Delta$ such that $v_n - \bar{g}_n \to v_0$. For sufficiently large $n$, $|||v_n - g_n - v_0||| < \operatorname{dist}(x - v_0, G) - \alpha$ and so $|||x - v_0 - g_n - \bar{g}_n||| \leqslant |||x - g_n - v_n||| + |||v_n - v_0 - \bar{g}_n||| < \operatorname{dist}(x - v_0, G)$, which is not possible.

Now suppose $V_X$ is reflexive. By (4.14) it follows that $\{v_n + g_n\}_{n \in \Delta}$ is bounded in $V_X$ and so we can suppose (passing to a subnet) that $v_n + g_n \rightharpoonup v$. Then $\hat{v}_n \rightharpoonup \hat{v}$ and since $\hat{v}_n \rightharpoonup \hat{v}_0$ it follows $v = v_0 + g$ for some $g \in G$. Then $|||x - v_0 - g||| \leqslant \liminf |||x - v_n - g_n||| \leqslant \alpha < \operatorname{dist}(x - v_0, G)$, which is not possible. Therefore $X/G$ is a nals if $\dim V_X/G < \infty$ or $V_X$ is reflexive. We do not know whether these assumptions are or not superfluous.

## 5. Strong Normed Almost Linear Spaces and Applications

Let $X$ be a nals and suppose there exists a semi-metric $\rho$ on $X$ (i.e., $\rho$ satisfies all the axioms of a metric except for $\rho(x, y) = 0$ implies $x = y$, $x, y \in X$), which satisfies $(M_1)$–$(M_3)$ below. Let $x, y, z \in X$ and $\lambda \in R$. $(M_1)$ $|\,|||x||| - |||y|||\,| \leqslant \rho(x, y) \leqslant |||x - y|||$; $(M_2)$ $\rho(x + z, y + z) \leqslant \rho(x, y)$; $(M_3)$ The function $\lambda \to \rho(\lambda x, x)$ is continuous at $\lambda = 1$.

5.1. Definition. A *strong normed almost linear space* (snals) is a nals $X$ together with a semi-metric $\rho$ satisfying $(M_1)$–$(M_3)$.

5.2. (a) *For $x, y \in X$ and $v \in V_X$ we have*

$$\rho(x + v, y + v) = \rho(x, y), \tag{5.1}$$

$$\rho(x, v) = |||x - v|||. \tag{5.2}$$

*Indeed, by* $(M_2)$ *we have* $\rho(x + v, y + v) \leqslant \rho(x, y) = \rho((x + v) - v, (y + v) - v) \leqslant \rho(x + v, y + v)$, *i.e.,* (5.1). *Hence* $\rho(x, v) = \rho(x - v, 0)$ *and by* $(M_1)$, $|||x - v||| \leqslant \rho(x - v, 0) \leqslant |||x - v|||$, *i.e., we have* (5.2). (b) *When $X$ is a* nls, *by* (5.2) *it follows that the only semi-metric satisfying* $(M_1)$–$(M_3)$ *is that generated by the norm.* (c) *By* (5.1), *to approximate $x \in X$ by elements of a subset $G \subset V_X$ in the norm $|||\cdot|||$ is the same as to approximate it in the semi-metric $\rho$. Since in this section we have also results concerning arbitrary $G \subset X$, we draw attention that in the sequel* $\operatorname{dist}(x, G)$ *and $P_G(x)$ are given by* (1.3) *and* (1.4) *for* $|||\cdot|||$.

We shall give now examples of snals.

5.3. EXAMPLES. (i) Let $X$ be the nals given in one of the Examples 3.2, 3.4, or 3.5. For $A_1, A_2 \in X$ let $\rho(A_1, A_2)$ be the Hausdorff semi-metric, i.e.,

$$\rho(A_1, A_2) = \max\{\sup_{a_1 \in A_1} \operatorname{dist}(a_1, A_2), \sup_{a_2 \in A_2} \operatorname{dist}(a_2, A_1)\}.$$

It is straightforward to show that $\rho$ satisfies $(M_1)$–$(M_3)$. Notice that in Examples 3.4 and 3.5, $\rho$ is a metric on $X$.

(ii) Let $X$ be the nals given in Example 3.6, and for $x_1, x_2 \in X$, let $\rho(x_1, x_2) = \|x_1 - x_2\|$ (here $x_1 - x_2$ is understanded in $E$). Since $E$ is an (AL)-space, we have $|\,|||x_1||| - |||x_2|||\,| = |\,\|x_1\| - \|x_2\|\,| \leqslant \|x_1 - x_2\| \leqslant \|x_1\| + \|x_2\| = |||x_1 + m(-1, x_2)|||$, i.e., $\rho$ satisfies $(M_1)$. It is obvious that it satisfies $(M_2)$ and $(M_3)$. Here $\rho$ is a metric on $X$.

(iii) Let $X$ be the nals given in Example 3.7, and for $x_1, x_2 \in X$, let $\rho(x_1, x_2) = |\varphi(x_1) - \varphi(x_2)|$. Then $\rho$ is a semi-metric on $X$ which satisfies $(M_1)$–$(M_3)$.

(iv) Let $X$ be the nals given in Example 3.11 or in Example 3.8 when $H$ is a semi $L$-summand in $E$. Then in both cases we have $|||x||| = \|x\|$ for each $x \in X$. For $x_1, x_2 \in X$ let $\rho(x_1, x_2) = \|x_1 - x_2\|$ (here $x_1 - x_2$ is understood in $E$). Then $\rho$ is a metric on $X$ satisfying $(M_1)$–$(M_3)$. Since $(M_2)$ and $(M_3)$ are obvious, we show $(M_1)$ for Example 3.8 (the proof for Example 3.11 being similar). Let $x_1, x_2 \in X$. Then $\varphi(x_i) = \varphi(x_i - p_H(x_i)) = \|x_i - p_H(x_i)\|$, $i = 1, 2$, and so $\rho(x_1, x_2) = \|x_1 - x_2\| \leqslant \|x_1 - p_H(x_1)\| + \|p_H(x_1) - p_H(x_2)\| + \|p_H(x_2) - x_2\| = \varphi(x_1 + x_2) + \|p_H(x_1) + p_H(x_2 - 2p_H(x_2))\| = \varphi(x_1 + m(-1, x_2)) + \|p_H(x_1 + m(-1, x_2))\| = |||x_1 + m(-1, x_2)|||$. For the other inequality in $(M_1)$ we use the assumption that $H$ is a semi $L$-summand in $E$. We have $|\,|||x_1||| - |||x_2|||\,| = |\,\|x_1\| - \|x_2\|\,| \leqslant \|x_1 - x_2\| = \rho(x_1, x_2)$.

Some other examples of snals can be obtained using Theorems 5.4 and 5.7. The first one states that the dual space of a nals (not necessarily a snals) is always a snals, and the second one states that when $X$ is a snals and $G$ a closed linear subspace of $V_X$, then in the space $X/G$ (not necessarily satisfying $(N_4)$) there exists a semi-metric satisfying $(M_1)$–$(M_3)$.

5.4. THEOREM. *For any* nals $X$, *the dual space* $X^*$ *is a* snals *for the metric* $\rho$ *defined by*

$$\rho(f_1, f_2) = \sup\{|f_1(x) - f_2(x)| : x \in B_X\} \qquad (f_1, f_2 \in X^*)$$

*Proof.* Clearly $\rho$ is a metric on $X$. To prove $(M_1)$, let $f_1, f_2 \in X^*$ and $x \in B_X$. Then $|f_1(x)| \leqslant |f_1(x) - f_2(x)| + |f_2(x)| \leqslant \rho(f_1, f_2) + |||f_2|||$, and since $x \in B_X$ was arbitrary, it follows $|||f_1||| \leqslant \rho(f_1, f_2) + |||f_2|||$. Similarly $|||f_2||| \leqslant \rho(f_1, f_2) + |||f_1|||$, whence the first inequality in $(M_1)$ follows. For the other inequality, let $x \in B_X$. By (2.6) we have that $f_1(x) - f_2(x) \leqslant$

$f_1(x)+f_2(-x) = (f_1+(-1\circ f_2))(x) \leqslant |||f_1+(-1\circ f_2)|||$. Similarly $f_2(x)-f_1(x)\leqslant |||f_1+(-1\circ f_2)|||$. Hence for each $x\in B_X$ we have $|f_1(x)-f_2(x)|\leqslant |||f_1+(-1\circ f_2)|||$ whence the right-hand side inequality in $(M_1)$ follows.

To prove $(M_2)$, let $f_i\in X^*$, $i=1, 2, 3$. Then $\rho(f_1+f_3, f_2+f_3)=\sup\{|(f_1+f_3)(x)-(f_2+f_3)(x)|: x\in B_X\}=\rho(f_1, f_2)$.

Finally we prove that for each $f\in X^*$, the function $\lambda\to\rho(\lambda\circ f, f)$ is continuous at any $\lambda>0$. Indeed, for $\lambda>0$ we have $\rho(\lambda\circ f, f)=\sup\{|f(\lambda x)-f(x)|: x\in B_X\}=|\lambda-1|\,|||f|||$. Therefore $X^*$ is a snals, which completes the proof.

In the sequel the folowing result will be of use.

5.5. LEMMA. *Let $X$ be a* snals, $\varnothing\neq G\subset X$ *and* $x, y\in X$. *We have*

$$|\text{dist}(x, G)-\text{dist}(y, G)|\leqslant\rho(x, y).$$

*Proof.* Let $\varepsilon>0$ be given and let $g\in G$ such that $|||y-g|||\leqslant \text{dist}(y, G)+\varepsilon$. By $(M_1)$ and $(M_2)$ we get $\text{dist}(x, G)-\text{dist}(y, G)\leqslant |||x-g|||-|||y-g|||+\varepsilon\leqslant\rho(x-g, y-g)+\varepsilon\leqslant\rho(x, y)+\varepsilon$. Similarly, $\text{dist}(y, G)-\text{dist}(x, G)\leqslant\rho(x, y)+\varepsilon$, whence $|\text{dist}(x, G)-\text{dist}(y, G)|\leqslant\rho(x, y)+\varepsilon$, and the lemma follows.

5.6. COROLLARY. *Let $X$ be a* snals, $G\subset X$ *and* $x, y\in X$ *such that* $\rho(x, y)=0$. *Then* $\text{dist}(x, G)=\text{dist}(y, G)$ *and* $P_G(x)=P_G(y)$.

*Proof.* By Lemma 5.5 we get $\text{dist}(x, G)=\text{dist}(y, G)$. Now let $g\in P_G(x)$. Then $0\leqslant |||y-g|||-\text{dist}(y, G) = |||y-g|||-\text{dist}(x, G) = |||y-g|||-|||x-g|||\leqslant\rho(y-g, x-g)\leqslant\rho(y, x)=0$ and so $g\in P_G(y)$, i.e., $P_G(x)\subset P_G(y)$. The other inclusion is proved in a similar way.

5.7. THEOREM. *Let $X$ be a* snals *with the semi-metric $\rho$ and $G$ a closed linear subspace of $V_X$. Then $\hat\rho$ defined for $\hat x_1, \hat x_2\in X/G$ by*

$$\hat\rho(\hat x_1, \hat x_2)=\inf_{g\in G}\rho(x_1+g, x_2)\qquad (x_i\in\hat x_i, i=1, 2)\tag{5.3}$$

*is a semi-metric on $X/G$ satisfying* $(M_1)$–$(M_3)$.

*Proof.* We first observe that since $\rho$ satisfies (5.1), the definition of $\hat\rho(\hat x_1, \hat x_2)$ in (5.3) does not depend on the choice of $x_i\in\hat x_i$. Clearly if $\hat x_1=\hat x_2$, then for $x\in\hat x_i$, $i=1, 2$ we have by (5.3) $\hat\rho(\hat x_1, \hat x_2)\leqslant\rho(x, x)=0$, i.e., $\hat\rho(\hat x_1, \hat x_2)=0$. By (5.1) for $\rho$ we get $\hat\rho(\hat x_1, \hat x_2)=\hat\rho(\hat x_2, \hat x_1)$. Now let $\hat x_i\in X/G$ and let $x_i\in\hat x_i$, $i=1, 2, 3$. Then for any $g_1, g_2\in G$ we have $\hat\rho(\hat x_1, \hat x_3)\leqslant\rho(x_1+g_1, x_3+g_2)\leqslant\rho(x_1+g_1, x_2)+\rho(x_2, x_3+g_2)$, whence since $g_1, g_2$ were arbitrary in $G$, it follows that $\hat\rho(\hat x_1, \hat x_3)\leqslant\hat\rho(\hat x_1, \hat x_2)+\hat\rho(\hat x_2, \hat x_3)$. Therefore $\hat\rho$ is a semi-metric on $X/G$.

To show $(M_1)$ for $\hat{\rho}$, let $\hat{x}_i \in X/G, x_i \in \hat{x}_i$, $i=1,2$, and $g \in G$. By Lemma 5.5 we have $| |||\hat{x}_1||| - |||\hat{x}_2||| | = |\text{dist}(x_1, G) - \text{dist}(x_2, G)| = |\text{dist}(x_1+g, G) - \text{dist}(x_2, G)| \leqslant \rho(x_1+g, x_2)$, whence since $g \in G$ was arbitrary, we get $| |||\hat{x}_1||| - |||\hat{x}_2||| | \leqslant \hat{\rho}(\hat{x}_1, \hat{x}_2)$. Now $\hat{\rho}(\hat{x}_1, \hat{x}_2) \leqslant \rho(x_1+g, x_2) \leqslant |||x_1 - x_2 + g|||$ and so $\hat{\rho}(\hat{x}_1, \hat{x}_2) \leqslant \text{dist}(x_1 - x_2, G) = |||\hat{x}_1 - \hat{x}_2|||$ which proves $(M_1)$.

To show $(M_2)$, let $\hat{x}_i \in X/G$, $x_i \in \hat{x}_i$, $i=1,2,3$, and $g \in G$. Then $\hat{\rho}(\hat{x}_1+\hat{x}_3, \hat{x}_2+\hat{x}_3) \leqslant \rho(x_1+x_3+g, x_2+x_3) \leqslant \rho(x_1+g, x_2)$, whence $(M_2)$ follows.

Finally, to show $(M_3)$ let $\hat{x} \in X/G$ and $x \in \hat{x}$. We have by (5.3), $\hat{\rho}(\lambda\hat{x}, \hat{x}) \leqslant \rho(\lambda x, x)$ and since $\rho(\lambda x, x) \to 0$ for $\lambda \to 1$, we obtain $(M_3)$ for $\hat{\rho}$, which completes the proof.

In a snals $X$ the semi-metric $\rho$ generates a topology on $X$ (which is not Hausdorff in general) and in the sequel when we shall say that a set is closed, open, etc., we shall understand that in this topology. If we need this topology to be Hausdorff, then we assume $\rho$ to be a metric on $X$. In view of 5.2(b), the topology on the nls $V_X$ generated by $\rho$ is the same as the topology generated by the norm $|||\cdot|||$.

5.8. *In a* nals $X$ *the set* $B_X(x, r)$ *is closed. Indeed, let* $\{x_n\}_{n=0}^{\infty} \subset X$ *such that* $|||x_n - x||| \leqslant r$, $n \geqslant 1$, *and* $\rho(x_n, x_0) \to 0$. *By* $(M_1)$ *and* $(M_2)$ *we have* $|||x_0 - x||| - |||x_n - x||| \leqslant \rho(x_0 - x, x_n - x) \leqslant \rho(x_0, x_n)$ *and so* $|||x_0 - x||| \leqslant r + \rho(x_0, x_n) \to r$.

5.9. *Let* $X$ *be a* snals *and* $G \subset X$. (a) *Let* $x \in X$, $y \in \text{Dom}(P_G)$ *and* $g \in P_G(y)$. *By* $(M_1)$, $(M_2)$ *and Lemma* 5.5 *it follows that*

$$|||x - g||| \leqslant \text{dist}(x, G) + 2\rho(y, x). \tag{5.4}$$

*Consequently, if* $\{x_n\}_{n=1}^{\infty} \subset \text{Dom}(P_G)$ *and* $x \in X$ *are such that* $\rho(x_n, x) \to 0$, *and* $g_n \in P_G(x_n)$, *then*

$$\lim |||x - g_n||| = \text{dist}(x, G). \tag{5.5}$$

(b) *For* $g_0 \in G$, *the set* $P_G^{-1}(g_0) = \{x \in X: g_0 \in P_G(x)\}$ *is closed. If* $G$ *is a linear subspace of* $V_X$, *then* $P_G^{-1}(g_0)$ *is a cone with vertex at* $g_0$.

In the framework of a snals $X$, we can discuss the continuity (semicontinuity) properties of the set-valued mapping $x \to P_G(x)$. That will be done from now on. The results are known either for best approximations in a nls or for best simultaneous approximations in a nls. We use the following abbreviations: uKsc for upper Kuratowski semicontinuous, usc (lsc) for upper (lower) semicontinuous. We draw attention to the fact that we use these semicontinuity properties in a slightly more general framework than in [25, 26], but the definitions are the same.

5.10. PROPOSITION. *Let $X$ be a* snals *and $G \subset X$.*

(i) *If $G$ is closed, then $P_G$ is* uKsc *at any $x \in \mathrm{Dom}(P_G)$.*

(ii) *If $G \subset V_X$, then $P_G$ is both* usc *and* lsc *at any $g \in G$.*

(iii) *If $G$ is a closed subset of $V_X$ and* span *$G$ is finite-dimensional, then $P_G$ is* usc *on $X$.*

*Proof.* (i) Let $x \in \mathrm{Dom}(P_G)$ and let $x_n \in X$ such that $\rho(x_n, x) \to 0$. Let $g_n \in P_G(x_n)$ and $g \in G$ with $\rho(g_n, g) \to 0$. By (5.5) we get $\lim |||x - g_n||| = \mathrm{dist}(x, G)$. By ($M_1$) and ($M_2$) we have $|||g - x||| - |||g_n - x||| \leqslant \rho(g - x, g_n - x) \leqslant \rho(g, g_n)$, whence $|||x - g||| = \mathrm{dist}(x, G)$, i.e., $g \in P_G(x)$.

(ii) Use Lemma 5.5.

(iii) By Proposition 4.2, $G$ is proximal in $X$. To show that $P_G$ is usc, use (i) above and the hypothesis on $G$.

We recall [15, p. 146] that a Banach space $B$ is an *E-space* if $B$ is reflexive, strictly convex, and $x_n, x \in S_B$, $x_n \rightharpoonup x$ imply $x_n \to x$.

5.11. THEOREM. *Let $X$ be a* snals *and suppose that $V_X$ is an E-space. Let $G$ be a closed linear subspace of $V_X$. Then $P_G$ is both* lsc *and* usc *at any $x \in R_X(G)$.*

*Proof.* By Proposition 4.2, $G$ is proximinal in $X$. Let $x \in R_X(G)$. By Theorem 4.6, $P_G(x)$ is a singleton, say, $P_G(x) = \{g_0\}$. By Proposition 5.10(ii) we can suppose $x \notin G$. Let $\{x_n\}_{n=1}^{\infty} \subset X$ such that $\rho(x_n, x) \to 0$, and let $g_n \in P_G(x_n)$. Then the sequence $\{g_n\}_{n=1}^{\infty}$ is bounded in $V_X$, since $|||g_n||| \leqslant 2\,|||x_n||| = 2\rho(x_n, 0) \leqslant 2(\rho(x_n, x) + \rho(x, 0))$. Since $V_X$ is reflexive and $P_G(x) = \{g_0\}$, by $N_4$ and (5.5) we get $g_n \rightharpoonup g_0$. By hypothesis, $x \in R_X(G)$ and so for $g_0 \in G$ there exists $v_0 \in V_X$ such that

$$|||x - g_0||| = |||v_0 - g_0|||, \tag{5.6}$$

$$|||x - g_n||| \geqslant |||v_0 - g_n|||, \qquad n \in N. \tag{5.7}$$

Let $\{g_{n_i}\} \subset \{g_n\}$ with $\lim\sup |||v_0 - g_n||| = \lim |||v_0 - g_{n_i}|||$. We have $v_0 - g_n \rightharpoonup v_0 - g_0$, and by (5.6), (5.7), and (5.5) we get $\mathrm{dist}(x, G) = |||x - g_0||| = |||v_0 - g_0||| \leqslant \lim\inf |||v_0 - g_n||| \leqslant \lim\sup |||v_0 - g_n||| = \lim |||v_0 - g_{n_i}||| \leqslant \lim\inf |||x - g_{n_i}||| = \mathrm{dist}(x, G)$. Therefore $|||v_0 - g_n||| \to |||v_0 - g_0|||$, and since $V_X$ is an $E$-space it follows that $v_0 - g_n \to v_0 - g_0$ and so $g_n \to g_0$. Since $g_n \in P_G(x_n)$ were arbitrary, it is now immediate that $P_G$ is both usc and lsc at $x \in R_X(G)$.

We conclude this paper mentioning that we have also extended for a snals, the following results from the theory of best (or best simultaneous) approximation in a nls: [13, Corollary 1; 14, Theorem 1; 17, Proposition; 20, Theorems 2 and 3; 21, Theorems 5 and 6]. The formulations and proofs will be given elsewhere.

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